The multiplicity of weights in nonprimitive pairs of weights

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Abstract

For each type of classical Lie algebra, we list the dominant highest weights ζ for which $(\zeta;\mu_i)$ is not a primitive pair and the weight space V_{μ_i} has dimension one where μ_i are the highest long and short roots in each case. These dimension one weight spaces lead to examples of nilmanifolds for which we cannot prove or disprove the density of closed geodesics.

1 Introduction

In our study [2] of the distribution of closed geodesics on nilmanifolds, we considered manifolds arising from a Lie group N with an associated Lie algebra $\mathfrak{N}$ constructed from an irreducible representation of a compact semisimple Lie algebra $\mathfrak{g}_0$ on a real finite dimensional vector space U. The nilmanifolds considered, $\Gamma\backslash N$, are those such that Γ arises from a Chevalley rational structure on $\mathfrak{N}$. The main result of that study classified such nilmanifolds as having the density of closed geodesics property if all roots of $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ were weights of $V = U^{\mathbb{C}}$ with multiplicity greater than or equal to two.

In [2] we reduce the multiplicity question to $\mathfrak{g}$ simple, thus throughout this article, we will assume that $\mathfrak{g}$ is a complex simple Lie algebra with a fixed base Δ of positive simple roots determined by a Cartan subalgebra $\mathfrak{h}$. Let $V = V(\lambda)$ denote a finite dimensional irreducible $g-$module with highest weight λ. The multiplicity of a weight μ is defined to be the dimension of the weight space $V_\mu \subseteq V$, and is denoted $K_{\lambda,\mu}$. By standard results of Lie theory (cf [8]), each root of $\mathfrak{g}$ is conjugate to the highest short root μ_1 or the highest long root μ_2, thus we can consider only these roots when finding the dimension of the weight spaces of interest. The results of this paper provide for each classical Lie algebra type, all highest weights for which the highest short or long roots give rise to weight spaces of dimension one when the root and weight are nonprimitive pairs

as defined below. The case of primitive pairs is completely answered in [2]. Thus this paper provides many cases in [2] for which the density of closed geodesics cannot be shown with the traditional methods used. These exceptional cases provide examples for which the distribution of closed geodesics is unknown and still being investigated. They may provide unique examples of nilmanifolds satisfying necessary conditions, but not having the density of closed geodesics.

Definition 1.1. *For $\mathfrak{g}$ simple, define a pair $(\lambda;\mu)$ of weights in Λ^+ to be* primitive *if $(\lambda-\mu)$ written as the sum of simple roots has all positive integer coefficients.*

Thus the pair $(\lambda;\mu)$ is said to be *nonprimitive* if in the sum $\lambda-\mu$, at least one simple root has a zero coefficient. The weights for which the highest roots give rise to weight spaces of dimension one in the primitive pair case as discussed in [2] were found using Theorem 1.2 below which is also applied repeatedly in this article.

By [1] we will be able to reduce to the primitive case to find all weight spaces of dimension one. Thus the following result will be the basis of our determination of all dominant weights λ such that $K_{\lambda,\mu_i}=1$ for $i=1,2$. In the notation of [1], $\mathbb{Z}_+$ is the set of all nonnegative integers and $\{\omega_i\}$ is the set of fundamental dominant weights relative to Δ (found in Table 1, page 69 of [8]). Additionally, the partial ordering of weights $\lambda\succ\mu$ means that $\lambda-\mu$ is a linear combination of simple roots with nonnegative coefficients.

Theorem 1.2 ([1], Theorem 1.3)**.** *All primitive pairs $(\lambda;\mu)$ such that $K_{\lambda,\mu}=1$, up to isomorphism of Dynkin diagrams, are exhausted by the following list:*

1. A_n $(n\geq 1)$: $\lambda=l\omega_1$, $\mu=\sum_{1\leq i\leq n} a_i\omega_i$ *where* $a_i\in\mathbb{Z}_+$ *and*

   $(l-\sum_{1\leq i\leq n} ia_i)\in(n+1)\mathbb{N}$

2. B_n $(n\geq 2)$: $\lambda=l\omega_1$, $\mu=\sum_{1\leq i\leq n} a_i\omega_i$ *where* $a_i\in\mathbb{Z}_+$ *is even and*

   $(l-1)=\sum_{1\leq i\leq n-1} ia_i+\dfrac{na_n}{2}$

3. G_2: $\lambda=l\omega_2$, $\mu=a_1\omega_1+a_2\omega_2$ *where* $a_1,\ a_2\in\mathbb{Z}_+$, *and* $3l-1=2a_1+3a_2$

4. G_2: $\lambda=\omega_1$, $\mu=0$.

For each $\mathfrak{g}$ of classical type, we first identify those highest dominant weights λ having the property that $\lambda\neq\mu_i$ and $(\lambda;\mu_i)$ is not a primitive pair for μ_i, the highest short or long root $i=1,2$ respectively. Once we have identified such λ, we reduce to a primitive pair by another result of [1] and then use Theorem 1.2 to determine whether $K_{\lambda,\mu_i}=1$. For review, we list the highest short and long roots in Table 1[8].

The remainder of this paper is the proof of the following result, considering case by case each class of simple Lie algebra.

Table 1: Highest short and long roots

Classical Lie algebra type	
A_n	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$
B_n	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$ $\alpha_1 + 2\alpha_2 \cdots + 2\alpha_n$
C_n	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n$ $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n$
D_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$2\alpha_1 + \alpha_2$ $3\alpha_1 + 2\alpha_2$

Theorem 1.3. *Consider $\mu = \mu_i$, $i = 1, 2$, the highest short and long roots of $\mathfrak{g}$. All nonprimitive pairs $(\zeta; \mu_i)$ such that $K_{\zeta,\mu_i} = 1$, up to isomorphism of Dynkin diagrams are exhausted by the Table 2.*

Table 2: Theorem 1.3

Lie algebra type	dominant highest weight ζ
A_n	$\alpha_1 + 2\alpha_2$, $2\alpha_1 + \alpha_2$ $(n = 2)$ $\alpha_1 + 2\alpha_2 + \alpha_3$ $(n = 3)$
B_n $(n \geq 2)$	$\alpha_1 + 2\alpha_2 + m_3\alpha_3$, $m_3 \geq 3$ $(n = 3)$ $2\alpha_1 + 2\alpha_2 \cdots + 2\alpha_n$ $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + \cdots + 3\alpha_n$ $(n \geq 4)$
D_n $(n \geq 4)$	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$, $\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4$ $(n = 4)$
G_2	$4\alpha_1 + 2\alpha_2$

Before continuing, a few more definitions and the following result are necessary. For any subset S of the set of simple roots Δ, define $\mathfrak{g}(S)$ to be the subalgebra of $\mathfrak{g}$ generated by the root subspaces $\mathfrak{g}_\beta$ and $\mathfrak{g}_{-\beta}$ for all $\beta \in S$. We define the projection map $p = p_S$ to be the natural projection of the set of weights of $\mathfrak{g}$ to the set of weights of $\mathfrak{g}(S)$. The Lie algebra $\mathfrak{g}(S)$ is known to be semisimple.

Proposition 1.4. *[1, Proposition 2.4]*

1. *Let S be a subset of simple roots. Let $\lambda \in \Lambda$ be an element such that the expansion of the weight $(\lambda - \mu)$ in terms of simple roots involves only elements of S. Then $K_{\lambda,\mu} = K_{p(\lambda),p(\mu)}$.*

2. *Under the assumptions of part 1, let $S_1, \ldots, S_k$ be all the connected components of the set S in the Dynkin diagram of the system of positive roots, and let $\lambda_i = p_{S_i}(\lambda)$ and $\mu_i = p_{S_i}(\mu)$. Then $K_{\lambda,\mu} = \Pi_{1\leq i\leq k} K_{\lambda_i,\mu_i}$.*

In our results, since we have reduced to considering $\mathfrak{g}$ simple, let S be the set of simple roots that occur with nonzero coefficients in the difference $\lambda - \mu$ and let $p = p_S$. We can then apply Proposition 1.4 and Theorem 1.2 to determine if $K_{p(\lambda),p(\mu)} = 1$. Thus for each Lie algebra and each highest weight λ where $(\lambda; \mu)$ is not a primitive pair, we must identify S and then determine $\mathfrak{g}(S)$. To find $\mathfrak{g}(S)$, we can simply consider the Dynkin diagram of the root system of S.

In many cases $S = \{\alpha_i\}$ and then $\mathfrak{g}(S) \cong A_1$. Thus the following lemma will be used frequently as we continue our discussion of nonprimitive pairs.

Lemma 1.5. *Let $\mathfrak{g}$ be of type A_1 with root α_1. Then $K_{2\alpha_1,\alpha_1} = 1$.*

Proof. Let $\lambda = 2\alpha_1$ and $\mu = \alpha_1$. By Theorem 1.2 for the case A_1, $K_{\lambda,\mu} = 1$ if $\lambda = l\omega_1$, $\mu = a_1\omega_1$ with $a_1 \in \mathbb{Z}_+$ and $l - a_1 \in 2\mathbb{N}$. Since $\mu = \alpha_1 = 2\omega_1$, $a_1 = 2$ and since $\lambda = 2\alpha_1 = 4\omega_1$, $l = 4$. Clearly then $l - a_1 \in 2\mathbb{N}$ and $K_{\lambda,\mu} = 1$. □

In each case of classical Lie algebra, we assume the zero weight space of our representation V is nontrivial, as necessary in [2]. Therefore every dominant weight λ is of the form $\lambda = \sum m_i\alpha_i$ where m_i must satisfy inequalities that arise from the conditions $\langle \lambda, \alpha_i \rangle \geq 0$ for $i = 1, \ldots, n$. Also $m_i > 0$ for all i by the following lemma of [2].

Lemma 1.6. *Let $\mu \in \Lambda^+$ and suppose that $\mu = \sum_{i=1}^{n} m_i\alpha_i$ for integers $\{m_i\}$. Then $m_k > 0$ for all k.*

This lemma is used in the proof of the following necessary proposition, found in [2].

Proposition 1.7. *Let V be an irreducible $\mathfrak{g}-$module with nontrivial zero weight space V_0. Let $\lambda \in \Lambda(V)^+$ be the highest weight. Then*

1. *$\lambda = \sum_{i=1}^{n} p_i\alpha_i$ for suitable positive integers p_i.*

2. *If $\mu \in \Lambda(V)$, then $\mu = \sum_{i=1}^{n} m_i\alpha_i$, $m_i \in \mathbb{Z}$. Furthermore, if $\mu \in \Lambda^+(V)$, then the integers $\{m_i\}$ are all positive.*

3. *At least one root of $\mathfrak{g}$ is a weight.*

2 Nonprimitive pairs for A_n

In this case, recall that $\mu_1 = \mu_2 = \mu = \alpha_1 + \cdots + \alpha_n$ is the dominant weight that is conjugate to all roots. We will find all dominant weights ζ such that $\zeta \succ \mu$ and $(\zeta; \mu)$ is not a primitive pair. Next we determine for which of these weights $K_{\zeta,\mu} = 1$.

Lemma 2.1. *For $\mathfrak{g}$ of type A_n with $n \geq 4$, there are no highest weights ζ such that $\zeta \succ \mu$, $(\zeta; \mu)$ is not a primitive pair and $K_{\zeta,\mu} = 1$. For $n = 2, 3$ the following exceptional cases occur such that $K_{\zeta_i,\mu} = 1$.*

1. $n = 2 \quad \zeta_1 = \alpha_1 + 2\alpha_2$, $\zeta_2 = 2\alpha_1 + \alpha_2$
2. $n = 3 \quad \zeta_3 = \alpha_1 + 2\alpha_2 + \alpha_3$

Lemma 2.2. *If S is a set of k consecutive simple roots in A_n, then $\mathfrak{g}(S) \cong A_k$.*

Proof. The rank of S is k, and the Cartan matrix or Dynkin diagram of the root system is the same as that of A_k, thus proving the claim. □

of Lemma 2.1. First we consider the cases $n = 2, 3$ and then show the general result for $n \geq 4$.

Case $\mathbf{n = 2}$

The weight $\zeta = m_1\alpha_1 + m_2\alpha_2$ is a dominant weight if and only if the following inequalities hold:

$$m_2 \leq 2m_1 \tag{1}$$
$$m_1 \leq 2m_2 \tag{2}$$

For the highest long root $\mu = \alpha_1 + \alpha_2$, let ζ be a highest weight such that $\zeta \succ \mu$ and $(\zeta; \mu)$ is not primitive. Then either (a) $m_1 = 1$ or (b) $m_2 = 1$.

(a) First let $m_1 = 1$. Then $m_2 = 1$ or $m_2 = 2$ by the inequalities above. If $m_2 = 1$, then $\zeta = \mu$, which is ruled out. Let $m_2 = 2$; then $\zeta = \zeta_1 = \alpha_1 + 2\alpha_2$. We find $\zeta_1 - \mu = \alpha_2$ and thus $S = \{\alpha_2\}$ and $\mathfrak{g}(S) \cong A_1$. Relabeling α_2 as α_1, the projection $p : \mathfrak{g} \to \mathfrak{g}(S)$ gives $p(\zeta_1) = 2\alpha_1$ and $p(\mu) = \alpha_1$. Then by Lemma 1.5 $K_{\zeta_1,\mu} = K_{p(\zeta_1),p(\mu)} = 1$.

(b) Similarly, we find that if $m_2 = 1$, then either $\zeta = \mu$, which is ruled out, or $\zeta = \zeta_2 = 2\alpha_1 + \alpha_2$. In the second case $\zeta_2 - \mu = \alpha_1$, and we conclude that $S = \{\alpha_1\}$ and $\mathfrak{g}(S) \cong A_1$. Since $p(\zeta_2) = 2\alpha_1$ and $p(\mu) = \alpha_1$, we again conclude that $K_{\zeta_2,\mu} = 1$ by 1.5.

Case $\mathbf{n = 3}$

A weight ζ is a dominant weight if and only if the following inequalities hold.

$$m_2 \leq 2m_1 \tag{1}$$
$$m_1 + m_3 \leq 2m_2 \tag{2}$$
$$m_2 \leq 2m_3 \tag{3}$$

In this case $\mu = \alpha_1 + \alpha_2 + \alpha_3$. Let ζ be a highest weight such that $(\zeta; \mu)$ is not primitive. Then $\zeta = m_1\alpha + m_2\alpha_2 + m_3\alpha_3$ and one of the following must be true: (a) $m_1 = 1, (b)\ m_2 = 1$, or (c) $m_3 = 1$. We will first find all such ζ and then determine if any give $K_{\zeta,\mu} = 1$.

(a) If $m_1 = 1$, then $\zeta = \alpha_1 + m_2\alpha_2 + m_3\alpha_3$ where m_2 and m_3 satisfy the above inequalities. By (1), $m_2 = 1$ or $m_2 = 2$. If $m_2 = 1$ we use (2) to conclude that $m_3 = 1$ and thus $\zeta = \mu$, which is ruled out. If $m_2 = 2$ then inequality (2) gives $m_3 \leq 3$ resulting in the following dominant weights ζ such that$(\zeta; \mu)$ is a nonprimitive pair:

$$\begin{aligned} \zeta = \zeta_1 &= \alpha_1 + 2\alpha_2 + \alpha_3 \\ \zeta = \zeta_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \zeta = \zeta_3 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 \end{aligned}$$

(b) Next consider $m_2 = 1$. By (2), $m_1 = m_3 = 1$ also and $\zeta = \mu$, which is ruled out.

(c) Finally, let $m_3 = 1$. By (3), $m_2 = 1$ or $m_2 = 2$. Again, if $m_2 = 1$, then $\zeta = \mu$, which is ruled out. If $m_2 = 2$, by (2) $m_1 \leq 3$ and thus we are left with two additional dominant weights ζ such that $(\zeta; \mu)$ is not primitive:

$$\begin{aligned} \zeta = \zeta_4 &= 2\alpha_1 + 2\alpha_2 + \alpha_3 \\ \zeta = \zeta_5 &= 3\alpha_1 + 2\alpha_2 + \alpha_3 \end{aligned}$$

We now determine $K_{\zeta_i,\mu} = 1$ in each case.

1. $\zeta_1 - \mu = \alpha_2$. Thus $S = \{\alpha_2\}$ and $\mathfrak{g}(S) \cong A_1$. Relabeling α_2 as α_1 we obtain $p(\zeta_1) = 2\alpha_1$ and $p(\mu) = \alpha_1$. We conclude that $K_{\zeta_1,\mu} = 1$ by Lemma 1.5.

2. $\zeta_2 - \mu = \alpha_2 + \alpha_3$. Thus $S = \{\alpha_2, \alpha_3\}$ and $\mathfrak{g}(S) \cong A_2$. Relabeling $\{\alpha_2, \alpha_3\}$ as $\{\alpha_1, \alpha_2\}$ we obtain $p(\zeta_2) = 2\alpha_1 + 2\alpha_2$ and $p(\mu) = \alpha_1 + \alpha_2$. According to Theorem 1.2 if $K_{\zeta_2,\mu} = K_{p(\zeta_2),p(\mu)} = 1$, then $p(\zeta_2) = l\omega_1$ for some positive integer l. We would have $p(\zeta_2) = 2\alpha_1 + 2\alpha_2 = l\omega_1 = \frac{l}{3}(2\alpha_1 + \alpha_2)$. Each weight is written as the unique sum of simple roots with positive integer coefficients, so this is impossible. Thus $K_{\zeta_2,\mu} \neq 1$.

3. $\zeta_3 - \mu = \alpha_2 + 2\alpha_3$. Thus $S = \{\alpha_2, \alpha_3\}$ and $\mathfrak{g}(S) \cong A_2$. Relabeling $\{\alpha_2, \alpha_3\}$ as $\{\alpha_1, \alpha_2\}$ yields $p(\zeta_3) = 2\alpha_1 + 3\alpha_2$ and $p(\mu) = \alpha_1 + \alpha_2$. As above, this satisfies the conditions for $K_{\zeta_3,\mu} = 1$ by Theorem 1.2 if $p(\zeta_3) = 2\alpha_1 + 3\alpha_2 = \frac{l}{3}(2\alpha_1 + \alpha_2)$ for some positive integer l. However the argument of (2) shows that l does not exist. Thus $K_{\zeta_3,\mu} \neq 1$.

4. $\zeta_4 - \mu = \alpha_1 + \alpha_2$. Thus $S = \{\alpha_1, \alpha_2\}$ and $\mathfrak{g}(S) \cong A_2$. Then $p(\zeta_4) = 2\alpha_1 + 2\alpha_2$ and $p(\mu) = \alpha_1 + \alpha_2$ and we have the same conditions as for ζ_2. Thus the same conclusion holds; $K_{\zeta_4,\mu} \neq 1$.

5. $\zeta_5 - \mu = 2\alpha_1 + \alpha_2$. Thus $S = \{\alpha_1, \alpha_2\}$ and $\mathfrak{g}(S) \cong A_2$. Then $p(\zeta_5) = 3\alpha_1 + 2\alpha_2$ and $p(\mu) = \alpha_1 + \alpha_2$. As above, this satisfies the conditions for

$K_{\zeta_5,\mu} = K_{p(\zeta_5),(\mu)} = 1$ by Theorem 1.2 if $p(\zeta_5) = 3\alpha_1 + 2\alpha_2 = \frac{l}{3}(2\alpha_1 + \alpha_2)$ for some l. However, again there is no such l. Thus $K_{\zeta_5,\mu} \neq 1$.

Thus for $n = 3$, $\zeta = \alpha_1 + 2\alpha_2 + \alpha_3$ is the only highest weight such that $(\zeta;\mu)$ is a nonprimitive pair and $K_{\zeta,\mu} = 1$.

Case n $\geq$ 4

Now we consider the general case for $n \geq 4$. First we will show that if $\zeta \succ \mu$ and $(\zeta;\mu)$ is not a primitive pair for $\mu = \alpha_1 + \cdots + \alpha_n$, then ζ must have one of the following forms:

$$\begin{aligned}
\zeta = \zeta_1 &= \alpha_1 + m_2\alpha_2 + m_3\alpha_3 + \cdots + m_{n-1}\alpha_{n-1} + \alpha_n \\
\zeta = \zeta_2 &= \alpha_1 + m_2\alpha_2 + m_3\alpha_3 + \cdots + m_{n-1}\alpha_{n-1} + m_n\alpha_n \\
\zeta = \zeta_3 &= m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + \cdots + m_{n-1}\alpha_{n-1} + \alpha_n
\end{aligned}$$

where $m_i \geq 2$.

Then we will show that $K_{\zeta_i,\mu} \neq 1$ in each of these cases, allowing us to conclude that there are no weights ζ for $n \geq 4$ such that $(\zeta;\mu)$ is a nonprimitive pair and $K_{\zeta,\mu} = 1$.

For $\zeta = \sum m_i\alpha_i$ a dominant weight, the following inequalities must hold:

$$m_2 \leq 2m_1 \tag{1}$$

$$m_{i-1} + m_{i+1} \leq 2m_i, \; i = 2, \ldots, n-1 \tag{2}$$

$$m_{n-1} \leq 2m_n \tag{3}$$

Lemma 2.0.1. *Suppose that $m_i = 1$ for some i, with $i = 2, \ldots, n-1$. Then $m_i = 1$ for all $i = 1, \ldots, n$.*

Proof. Recall that $m_i \geq 1$ for all i by Proposition 1.7. If $m_i = 1$ for some $i = 2, \ldots, n-1$, then by inequality (2), $m_{i-1} + m_{i+1} \leq 2m_i = 2$, resulting in $m_{i-1} = m_{i+1} = 1$. By induction on (2), then $m_i = 1$ for all i. $\square$

By the result above, it is clear that $\{\zeta_1, \zeta_2, \zeta_3\}$ are the only dominant weights ζ different from μ for which $(\zeta;\mu)$ is not a primitive pair since $\mu = \alpha_1 + \cdots + \alpha_n$.

Next we show that $K_{\zeta_i,\mu} \neq 1$ in each case.

Case 1 $\zeta_1 = \alpha_1 + m_2\alpha_2 + \cdots + m_{n-1}\alpha_{n-1} + \alpha_n$, where $m_i \geq 2$ for $2 \leq i \leq n-1$.

From the difference $\zeta_1 - \mu = (m_2 - 1)\alpha_2 + \cdots + (m_{n-1} - 1)\alpha_{n-1}$ we see that $S = \{\alpha_2, \ldots, \alpha_{n-1}\}$ and then $\mathfrak{g}(S) \cong A_{n-2}$. Relabeling $\{\alpha_2, \ldots, \alpha_{n-1}\}$ as $\{\alpha_1, \ldots, \alpha_{n-2}\}$ we obtain $p(\zeta_1) = m_2\alpha_1 + m_3\alpha_2 + \cdots + m_{n-1}\alpha_{n-2}$ and $p(\mu) = \alpha_1 + \cdots + \alpha_{n-2}$. We see that $(p(\zeta_1);p(\mu))$ is now a primitive pair. By Theorem 1.2, $K_{\zeta_1,\mu} = K_{p(\zeta_1),p(\mu)} = 1$ if and only if $p(\zeta_1) = l\omega_1 = \frac{l}{n-1}((n-2)\alpha_1 + (n -$

$3)\alpha_2 + \cdots + \alpha_{n-2})$ for some l. If this is true, then

$$\begin{aligned} m_2 &= \frac{l(n-2)}{n-1} \\ m_3 &= \frac{l(n-3)}{n-1} \\ &\vdots \\ m_{n-2} &= \frac{2l}{n-1} \\ m_{n-1} &= \frac{l}{n-1} \end{aligned}$$

or equivalently

$$\begin{aligned} m_{n-2} &= 2m_{n-1} \\ m_{n-3} &= 3m_{n-1} \\ &\vdots \\ m_2 &= (n-2)m_{n-1} \end{aligned}$$

Thus we conclude that $m_i = (n-i)m_{n-1}$; i.e. $\{m_i\}_{i=2}^{n-1}$ is strictly decreasing. However, we have assumed that $m_1 = 1$, so by inequality (1), $m_2 \leq 2$. Then $m_3 < m_2$ means that $m_3 \leq 1$ which is a contradiction. Therefore there is no such dominant weight of the form ζ_1 such that $K_{\zeta_1,\mu} = 1$.

Case 2 $\zeta_2 = \alpha_1 + m_2\alpha_2 + \cdots + m_{n-1}\alpha_{n-1} + m_n\alpha_n$, where $m_i \geq 2$ for $2 \leq i \leq n-1$.

From the difference $\zeta_2 - \mu = (m_2 - 1)\alpha_2 + \cdots + (m_n - 1)\alpha_n$ we obtain $S = \{\alpha_2, \ldots, \alpha_n\}$ and then conclude that $\mathfrak{g}(S) \cong A_{n-1}$. Relabeling $\{\alpha_2, \ldots, \alpha_n\}$ as $\{\alpha_1, \ldots, \alpha_{n-1}\}$ we obtain $p(\zeta_2) = m_2\alpha_1 + m_3\alpha_2 + \cdots + m_n\alpha_{n-1}$ and $p(\mu) = \alpha_1 + \cdots + \alpha_{n-1}$. We see that $(p(\zeta_2); p(\mu))$ is now a primitive pair. By Theorem 1.2, $K_{\zeta_2,\mu} = K_{p(\zeta_2),p(\mu)} = 1$ if and only if $p(\zeta_2) = l\omega_1 = \frac{l}{n}((n-1)\alpha_1 + (n-2)\alpha_2 + \cdots + \alpha_{n-1})$ for some l. As above, this requirement allows us to conclude that $\{m_i\}_{i=2}^{n}$ is a decreasing sequence. However, we find the same contradiction as above, and therefore $K_{\zeta_2,\mu} \neq 1$.

Case 3 $\zeta_3 = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_{n-1}\alpha_{n-1} + \alpha_n$, where $m_i \geq 2$ for $2 \leq i \leq n-1$.

From the difference $\zeta_3 - \mu = (m_1 - 1)\alpha_1 + \cdots + (m_{n-1} - 1)\alpha_{n-1}$ we obtain $S = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and then conclude that $\mathfrak{g}(S) \cong A_{n-1}$. Then $p(\zeta_3) = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_{n-1}\alpha_{n-1}$ and $p(\mu) = \alpha_1 + \cdots + \alpha_{n-1}$. Again, $(p(\zeta_3); p(\mu))$ is now a primitive pair. By Theorem 1.2, $K_{p(\zeta_3),p(\mu)} = K_{\zeta_3,\mu} = 1$ if and only if $p(\zeta_3) = l\omega_1 = \frac{l}{n}((n-1)\alpha_1 + (n-2)\alpha_2 + \cdots + \alpha_{n-1})$ for some l. If this is true,

then the following set of equalities holds.

$$\begin{aligned} m_1 &= \frac{l}{n}(n-1) \\ m_2 &= \frac{l}{n}(n-2) \\ &\vdots \\ m_{n-1} &= \frac{l}{n} \end{aligned}$$

This is equivalent to

$$\begin{aligned} m_1 &= (n-1)m_{n-1} \\ &\vdots \\ m_{n-2} &= 2m_{n-1} \end{aligned}$$

By inequality (3) $m_{n-1} \leq 2m_n = 2$, we conclude that $m_{n-1} = 2$ since $m_{n-1} = 1$ implies $m_i = 1$ for all i by Sublemma 2.0.1. Then $m_i = 2(n-i)$ for all i and in particular $m_{n-2} = 4$. However, by (2), $m_{n-2} + m_n \leq 2m_{n-1}$ which implies that $4 + 1 \leq 4$, an obvious contradiction. Therefore there is no such dominant weight ζ_3 such that $K_{\zeta_3,\mu} = 1$, proving our claim. □

3 Nonprimitive pairs for $B_n,\ n \geq 2$

In this case, recall that the highest short and long roots are $\mu_1 = \alpha_1 + \cdots + \alpha_n$ and $\mu_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n$. Since we are restricting to the case that all roots are weights, we only need to find dominant weights ζ such that $\zeta \succ \mu_2$ and consider nonprimitive pairs $(\zeta;\mu_i)$ for $i = 1, 2$. Then we will determine for which of these weights $K_{\zeta,\mu_i} = 1$ for $i = 1$ or $i = 2$.

We recall a result of [2] for the B_n case:

Lemma 3.0.2. *Let $\lambda = m_1\alpha_1 + \cdots m_n\alpha_n \in \Lambda^+(V)$. If $m_1 = 1$, then either $m_i = 1$ for $1 \leq i \leq n$ or $m_i \geq 2$ for $2 \leq i \leq n$.*

Lemma 3.1. *For $\mathfrak{g}$ of type B_n, the only highest weights ζ such that $\zeta \succ \mu_2$, $(\zeta;\mu_i)$ is nonprimitive and $K_{\zeta,\mu_i} = 1$ for $i = 1$ or $i = 2$ are*

1. *$\zeta_1 = \alpha_1 + 2\alpha_2 + m_3\alpha_3$, where $m_3 \geq 3$*
2. *$\zeta_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + \cdots + 3\alpha_n$, $n \geq 4$*
3. *$\zeta_3 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$, $n \geq 2$*

Proof. First we consider the case $n = 2$ and then show the result for $n \geq 3$.
Case n = 2

The weight $\zeta = m_1\alpha_1 + m_2\alpha_2$ is a dominant weight if and only if the following inequalities hold:

$$\begin{aligned} m_2 &\leq 2m_1 \quad (1) \\ m_1 &\leq m_2 \quad (2) \end{aligned}$$

In this case $\mu_1 = \alpha_1 + \alpha_2$ and $\mu_2 = \alpha_1 + 2\alpha_2$. Let $\zeta = m_1\alpha_1 + m_2\alpha_2$ be a dominant weight such that $\zeta \succ \mu_2$ and $(\zeta; \mu_i)$ is not a primitive pair for $i = 1$ or $i = 2$. Hence $m_1 \geq 1$, $m_2 \geq 2$ and one of the following equalities must hold: (a) $m_1 = 1$ or (b) $m_2 = 2$.

(a) First consider $m_1 = 1$. By the inequalities above and the fact that $\zeta \succ \mu_2$, we have $m_2 = 2$. Thus $\zeta = \mu_2$, which is ruled out.

(b) Let $m_2 = 2$. By the inequalities above, $m_1 = 1$ or $m_1 = 2$. If $m_1 = 1$, we have case (a), so let $m_1 = 2$. Then $\zeta = 2\alpha_1 + 2\alpha_2$ and the pair $(\zeta; \mu_1)$ is primitive while $(\zeta; \mu_2)$ is nonprimitive. From the difference $\zeta - \mu_2 = \alpha_1$, we observe that $S = \{\alpha_1\}$ and $\mathfrak{g}(S) \cong A_1$. The projection $p : \mathfrak{g} \to \mathfrak{g}(S)$ gives $p(\zeta) = 2\alpha_1$ and $p(\mu_2) = \alpha_1$. Thus by Lemma 1.5, $K_{\zeta,\mu_2} = 1$. This is ζ_3 above for $n = 2$.

Case $\mathbf{n \geq 3}$

For $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$ a dominant weight, the following inequalities must hold:

$$\begin{aligned} m_2 &\leq 2m_1 \quad (1) \\ m_{i-1} + m_{i+1} &\leq 2m_i \text{ for } i = 2, \ldots, n-1 \quad (2) \\ m_{n-1} &\leq m_n \quad (3) \end{aligned}$$

We find all dominant weights $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$ such that $\zeta \succ \mu_2$ and $(\zeta; \mu_i)$ is not a primitive pair for either $i = 1$ or $i = 2$. If ζ is a dominant weight such that $(\zeta; \mu_1)$ is a nonprimitive pair, then $m_i = 1$ for some i. If ζ is a dominant weight such that $(\zeta; \mu_2)$ is a nonprimitive pair, then $m_1 = 1$ or $m_i = 2$ for $i = 2, \ldots, n$, or both. Once we have found all such ζ, we will then find K_{ζ,μ_i}.

Since $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n \succ \mu_2$ we have (*) $m_1 \geq 1$ and $m_i \geq 2$ for all $i = 2, \ldots, n$.

Lemma 3.0.3. *Let $m_i = 2$ for some $i \geq 3$, then $m_i = 2$ for $i = 2, \ldots, n$.*

Proof. By (2), $m_{i-1} + m_{i+1} \leq 2m_i = 4$. Then if $m_{i-1} > 2$ it must be that $m_{i-1} = 3$ and $m_{i+1} = 1$ which contradicts Sublemma 3.0.2. Thus $m_{i-1} = m_{i+1} = 2$. By induction on (2) then $m_i = 2$ for $i = 2, \ldots, n$. □

If $(\zeta; \mu_1)$ is a nonprimitive pair with $\zeta \succ \mu_2$, then $m_1 = 1$. By inequality (1) and the fact that $m_2 \geq 2$, we conclude that $m_2 = 2$. Then $\zeta = \alpha_1 + 2\alpha_2 + m_3\alpha_3 + \cdots + m_n\alpha_n$. If $m_i = 2$ for some $i = 3, \ldots, n$, then by the previous sublemma $m_i = 2$ for all $i = 3, \ldots, n$ which means that $\zeta = \mu_2$, which we have ruled out. Hence $m_i \geq 3$ for $i = 3, \ldots, n$. We have proved the following:

(a) If $(\zeta;\mu_1)$ is a nonprimitive pair with $\zeta \succ \mu_2$, then $\zeta = \alpha_1 + 2\alpha_2 + m_3\alpha_3 + \cdots + m_n\alpha_n$, where $m_i \geq 3$ for $i \geq 3$.

We turn our attention to μ_2 and find those dominant weights ζ such that $\zeta \succ \mu_2$ and $(\zeta;\mu_2)$ is not a primitive pair. We may assume that $m_1 \geq 2$ for if $m_1 = 1$ then ζ lies in the list (a) by the argument above.

If $m_1 \geq 2$, $\zeta \succ \mu_2$ and $(\zeta;\mu_2)$ is not a primitive pair, then $m_i = 2$ for some $i \geq 2$. If $i \geq 3$, then $\zeta = \mu_2$ by the sublemma above, but this is ruled out. Hence $m_2 = 2$ and from (2) it follows that $2 + m_3 \leq m_1 + m_3 \leq 2m_2 = 4$. This implies $m_3 \leq 2$ but $m_3 \geq 2$ since $\zeta \succ \mu_2$. We conclude that $m_3 = 2$ and $m_1 = 2$. From the sublemma above we obtain

(b) If $(\zeta;\mu_2)$ is a nonprimitive pair with $\zeta \succ \mu_2$, then either ζ lies in the list (a) or $\zeta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$.

Thus we conclude that if $\zeta \succ \mu_2$ is a dominant weight with $(\zeta;\mu_1)$ or $(\zeta;\mu_2)$ nonprimitive, then by (a) and (b) we have two cases to consider. In the second case we consider only $(\zeta;\mu_2)$.

$$\begin{aligned}
\zeta &= \alpha_1 + 2\alpha_2 + m_3\alpha_3 + \cdots + m_n\alpha_n,\ m_i \geq 3 \text{ for } i \geq 3 \\
\zeta &= 2\alpha_1 + \cdots + 2\alpha_n
\end{aligned}$$

Case 1 $\zeta = \alpha_1 + 2\alpha_2 + m_3\alpha_3 + \cdots + m_n\alpha_n$, where $m_i \geq 3$ for $i \geq 3$.

First consider the nonprimitive pair $(\zeta;\mu_1)$. From the difference $\zeta - \mu_1 = \alpha_2 + (m_3-1)\alpha_3 + \cdots + (m_n-1)\alpha_n$, we obtain $S = \{\alpha_2, \ldots, \alpha_n\}$ and thus $\mathfrak{g}(S) \cong B_{n-1}$ by comparing Dynkin diagrams. Relabeling $\{\alpha_2, \ldots, \alpha_n\}$ as $\{\alpha_1, \ldots, \alpha_{n-1}\}$ yields $p(\zeta) = 2\alpha_1 + m_3\alpha_2 + \cdots + m_n\alpha_{n-1}$ and $p(\mu_1) = \alpha_1 + \cdots + \alpha_{n-1}$. By Theorem 1.2, $K_{p(\zeta),p(\mu_1)} = K_{\zeta,\mu_1} = 1$ implies that $p(\mu_1) = \sum_{1 \leq i \leq n-1} a_i\omega_i$, where $a_i \in \mathbb{Z}_+$ is even. However, $p(\mu_1) = \omega_1$, so $a_1 = 1$ is not even. Therefore $K_{\zeta,\mu_1} \neq 1$.

We next consider the nonprimitive pair $(\zeta;\mu_2)$. From the difference $\zeta - \mu_2 = (m_3-2)\alpha_3 + \cdots + (m_n-2)\alpha_n$ we see that $S = \{\alpha_3, \ldots, \alpha_n\}$. We need to consider two subcases: $n = 3$ and $n \geq 4$.

Subcase $\mathbf{n = 3}$

Here $\mathfrak{g}(S) \cong A_1$. Relabeling $\{\alpha_3\}$ as $\{\alpha_1\}$, we obtain $p(\zeta) = m_3\alpha_1 = 2m_3\omega_1$ and $p(\mu_2) = 2\alpha_1 = 4\omega_1$. By the A_1 case of Theorem 1.2, $K_{p(\zeta),p(\mu_2)} = 1$ if $p(\zeta) = l\omega_1$ and $p(\mu_2) = a\omega_1$ where $l - a \in 2\mathbb{N}$. Thus μ_2 has multiplicity one if $2m_3 - 4 \in 2\mathbb{N}$, which holds for $m_3 \geq 2$. However, if $m_3 = 2$, then $\zeta = \mu_2$, which is ruled out. Hence for $m_3 \geq 3$, $(\zeta;\mu_2)$ is nonprimitive and $K_{\zeta,\mu_2} = 1$. Define $\zeta = \zeta_1$ in this case.

Subcase $\mathbf{n \geq 4}$

Here $\mathfrak{g}(S) \cong B_{n-2}$ since the Dynkin diagrams are the same. Relabeling $\{\alpha_3, \ldots, \alpha_n\} \to \{\alpha_1, \ldots, \alpha_{n-2}\}$ we obtain $p(\zeta) = m_3\alpha_1 + m_4\alpha_2 + \cdots + m_n\alpha_{n-2}$ and $p(\mu_2) = 2\alpha_1 + \cdots + 2\alpha_{n-2} = 2\omega_1$.

We use the B_n case of Theorem 1.2 to determine if $K_{p(\zeta),p(\mu_2)} = K_{\zeta,\mu_2} = 1$, namely if $p(\zeta) = l\omega_1$, $p(\mu_2) = \sum_{1 \leq i \leq n-2} a_i\omega_i$, where $a_i \in \mathbb{Z}_+$, even and $(l-1) = \sum_{1 \leq i \leq n-3} ia_i + (n-2)a_{n-2}/2$.

Since $p(\mu_2) = 2\omega_1$ it follows that $a_1 = 2$ and $a_i = 0$ for all other $i \neq 1$. Thus $l = 3$ and then for $K_{p(\zeta),p(\mu_2)} = 1$ we must have $p(\zeta) = l\omega_1 = 3\alpha_1 + \cdots + 3\alpha_{n-2}$; that is, $m_i = 3$ for all $i = 3, \ldots, n$. Thus in this case, the only dominant weight ζ such that $K_{\zeta,\mu_2} = 1$ is $\zeta = \zeta_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + \cdots + 3\alpha_n$.

Case 2 $\zeta = \zeta_3 = 2\alpha_1 + \cdots + 2\alpha_n$, $n \geq 3$.

From the difference $\zeta - \mu_2 = \alpha_1$ we observe that $S = \{\alpha_1\}$ and $\mathfrak{g}(S) \cong A_1$. Then $p(\zeta) = 2\alpha_1$ and $p(\mu_2) = \alpha_1$. By Lemma 1.5, $K_{p(\zeta),p(\mu_2)} = K_{2\alpha_1,\alpha_1} = 1$. □

4 Nonprimitive pairs for $C_n,\ n \geq 3$

In this case recall that $\mu_1 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$ and $\mu_2 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$ are the highest short and long roots.

Lemma 4.1. *For $\mathfrak{g}$ of type C_n, there are no highest weights ζ such that $\zeta \succ \mu_2$, $(\zeta; \mu_i)$ is a nonprimitive pair and $K_{\zeta,\mu_i} = 1$ for $i = 1$ or $i = 2$.*

Proof. Let us first consider the case $n = 3$ and then we will investigate the general case for $n \geq 4$.

Case n = 3

Here $\mu_1 = \alpha_1 + 2\alpha_2 + \alpha_3$ and $\mu_2 = 2\alpha_1 + 2\alpha_2 + \alpha_3$ and we let $\zeta = m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 \succ \mu_2$, which implies (*) $m_1 \geq 2$, $m_2 \geq 2$ and $m_3 \geq 1$. If $(\zeta; \mu_i)$ is a nonprimitive pair for $i = 1, 2$, then one of the following must hold: (a) $m_1 = 2$, (b) $m_2 = 2$ or (c) $m_3 = 1$. Recall first that ζ is a dominant weight if and only if the following inequalities hold.

$$\begin{aligned} m_2 &\leq 2m_1 & (1)\\ m_1 + 2m_3 &\leq 2m_2 & (2)\\ m_2 &\leq 2m_3 & (3) \end{aligned}$$

We consider each of the cases above to determine nonprimitive pairs.

(a) Suppose that $m_1 = 2$. Then by (1) $m_2 \leq 2m_1 = 4$ and by (*) $m_2 \geq 2$, so therefore $m_2 = 2, 3$ or 4.

(i) If $m_2 = 2$, then (2) gives $2 + 2m_3 = m_1 + 2m_3 \leq 2m_2 = 4$ and (3) gives $2 = m_2 \leq 2m_3$, together yielding $m_3 = 1$. Then $\zeta = \mu_2$, which is ruled out.

(ii) If $m_2 = 3$, then by (2) $2 + 2m_3 = m_1 + 2m_3 \leq 2m_2 = 6$ and by (3) $3 = m_2 \leq 2m_3$, giving $m_3 = 2$. Then $\zeta = 2\alpha_1 + 3\alpha_2 + 2\alpha_3$, a candidate.

(iii) If $m_2 = 4$, then by (2) $2 + 2m_3 = m_1 + 2m_3 \leq 2m_2 = 8$ and by (3) $4 = m_2 \leq 2m_3$. We conclude $2 \leq m_3 \leq 3$. Then $\zeta = 2\alpha_1 + 4\alpha_2 + 2\alpha_3$ or $\zeta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3$, both candidates.

(b) Suppose that $m_2 = 2$. By (2) $m_1 + 2m_3 \leq 2m_2 = 4$ and by (*), $m_1 \geq 2$ and $m_3 \geq 1$, forcing the inequalities to be equalities. We conclude that $\zeta = \mu_2$, which is ruled out.

(c) If $m_3 = 1$, then by (3) $m_2 \leq 2m_3 = 2$ and by (*) $m_2 \geq 2$. Thus $m_2 = 2$. By (2) we see that $m_1 \leq 2$ and equality holds by (*). Again we find that $\zeta = \mu_2$, which is ruled out.

Note that if ζ is a dominant weight such that $\zeta \succ \mu_2$, then $(\zeta; \mu_1)$ is a primitive pair by (*) and cases (b) and (c) above. By the discussion above the only dominant weights ζ such that $\zeta \succ \mu_2$ and $(\zeta; \mu_2)$ is not a primitive pair are

$$\begin{aligned}
\zeta &= \zeta_1 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 \\
\zeta &= \zeta_2 = 2\alpha_1 + 4\alpha_2 + 2\alpha_3 \\
\zeta &= \zeta_3 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3
\end{aligned}$$

We will show for $i = 1, 2, 3$, that $K_{\zeta_i,\mu_2} \neq 1$.

From the difference $\zeta_1 - \mu_2 = \alpha_2 + \alpha_3$, we obtain $S = \{\alpha_2, \alpha_3\}$ and $\mathfrak{g}(S) \cong B_2$. Then relabeling $\{\alpha_2, \alpha_3\}$ as $\{\alpha_2, \alpha_1\}$ yields $p(\zeta_1) = 2\alpha_1 + 3\alpha_2$ and $p(\mu_2) = \alpha_1 + 2\alpha_2$. By Theorem 1.2, for a Lie algebra of type B_2, $K_{\zeta_1,\mu_2} = K_{p(\zeta_1),p(\mu_2)} = 1$ if and only if $p(\zeta_1) = l\omega_1$ and $p(\mu_2) = a_1\omega_1 + a_2\omega_2$ where $a_i \in \mathbb{Z}_+$ are even and $(l-1) = a_1 + a_2$. In our case, $p(\mu_2) = 2\omega_2$ and hence $l = 3$. Then for μ_2 to have multiplicity one, $p(\zeta_1) = 2\alpha_1 + 3\alpha_2 = 3\omega_1 = 3(\alpha_1 + \alpha_2)$, which is false. Therefore $K_{\zeta_1,\mu_2} \neq 1$.

From the difference $\zeta_2 - \mu_2 = 2\alpha_2 + \alpha_3$, we obtain $S = \{\alpha_2, \alpha_3\}$ and $\mathfrak{g}(S) \cong B_2$ as above. Then relabeling $\{\alpha_2, \alpha_3\}$ as $\{\alpha_2, \alpha_1\}$ yields $p(\zeta_2) = 2\alpha_1 + 4\alpha_2$ and $p(\mu_2) = \alpha_1 + 2\alpha_2$. We are in the same case of Theorem 1.2 as for ζ_1 where $l = 3$. Then for μ_2 to have multiplicity one, $p(\zeta_2) = 2\alpha_1 + 4\alpha_2 = 3\omega_1 = 3(\alpha_1 + \alpha_2)$, which is again false. Therefore $K_{\zeta_2,\mu_2} \neq 1$.

From the difference $\zeta_3 - \mu_2 = 2\alpha_2 + 2\alpha_3$, we obtain $S = \{\alpha_2, \alpha_3\}$ and $\mathfrak{g}(S) \cong B_2$ as above. Then relabeling $\{\alpha_2, \alpha_3\}$ as $\{\alpha_2, \alpha_1\}$ yields $p(\zeta_3) = 3\alpha_1 + 4\alpha_2$ and $p(\mu_2) = \alpha_1 + 2\alpha_2$. Again, we are in the same case of 1.2 as for ζ_1 where $l = 3$. Then for μ_3 to have multiplicity one, $p(\zeta_3) = 3\alpha_1 + 4\alpha_2 = 3\omega_1 = 3(\alpha_1 + \alpha_2)$, which is again false. Therefore $K_{\zeta_3,\mu_2} \neq 1$ also.

Thus, in the case $n = 3$, there are no dominant weights ζ such that $(\zeta; \mu_2)$ is a nonprimitive pair and $K_{\zeta,\mu_2} = 1$.

Case n $\geq$ 4

If $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$ is a dominant weight, then the following inequalities must hold.

$$\begin{aligned}
m_2 &\leq 2m_1 && (1) \\
m_{i-1} + m_{i+1} &\leq 2m_i,\ i = 2, \ldots, n-2 && (2) \\
m_{n-2} + 2m_n &\leq 2m_{n-1} && (3) \\
m_{n-1} &\leq 2m_n && (4)
\end{aligned}$$

Since we also assume $\zeta \succ \mu_2$, the following inequalities must hold as well: (*) $m_i \geq 2$ for $i = 1, \ldots, n-1$ and $m_n \geq 1$. For ζ such that $(\zeta; \mu_i)$ is not a primitive pair, $i = 1, 2$, at least one of these inequalities must be an equality.

Lemma 4.0.4. *If $m_i = 2$ for some $i = 2, \ldots, n-1$, then $m_i = 2$ for all $i = 1, \ldots, n-1$ and $m_n = 1$.*

Proof. Suppose that $m_i = 2$ for some $i = 2, \ldots, n-2$. Then by inequality (2) and (*), $m_{i-1} = m_{i+1} = 2$. We continue by induction on (2) and find that $m_i = 2$ for $i = 1, \ldots, n-1$. If $m_{n-1} = 2$, then by (3) $m_{n-2} + 2m_n \leq 2m_{n-1} = 4$. By (*) we know that $m_{n-2} \geq 2$ and $m_n \geq 1$, and it follows that $m_{n-2} = 2$ and $m_n = 1$. We now apply the first part of the argument to conclude that $m_i = 2$ for $i = 1, \ldots, n-1$. ☐

If ζ is a dominant weight such that $\zeta \succ \mu_2$, then we show that $(\zeta; \mu_1)$ is primitive and $(\zeta; \mu_2)$ is nonprimitive only if $m_1 = 2$. By Sublemma 4.0.4 if $m_i = 2$ for some $i = 2, \ldots, n-1$, then $m_i = 2$ for $i = 1, \ldots, n-1$ and $m_n = 1$. It follows that $\zeta = \mu_2$, which is ruled out. Similarly, if $m_n = 1$, then by inequality (4) and (*) we obtain $m_{n-1} = 2$. From Sublemma 4.0.4 we conclude that $\zeta = \mu_2$, which is ruled out. Hence $(\zeta; \mu_1)$ is a primitive pair. Thus the only case left to consider is when $m_1 = 2$.

If $\zeta \succ \mu_2$ and $(\zeta; \mu_2)$ is nonprimitive, then the previous paragraph shows that $\zeta = 2\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n$ where $m_i \geq 3$ for $i = 2, \ldots, n-1$ and $m_n \geq 2$. We show that $K_{\zeta,\mu_2} \neq 1$.

From the difference $\zeta - \mu_2 = (m_2 - 2)\alpha_2 + \cdots + (m_{n-1} - 2)\alpha_{n-1} + (m_n - 1)\alpha_n$ we find that $S = \{\alpha_2, \ldots, \alpha_n\}$ and $\mathfrak{g}(S) \cong C_{n-1}$ since $n \geq 4$. Relabeling $\{\alpha_2, \ldots, \alpha_n\}$ as $\{\alpha_1, \ldots, \alpha_{n-1}\}$ we find $p(\zeta) = m_2\alpha_1 + \cdots + m_n\alpha_{n-1}$ and $p(\mu_2) = 2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1}$. Thus $(p(\zeta); p(\mu_2))$ is a primitive pair for a Lie algebra of type C_{n-1}. By Theorem 1.2, there are no primitive pairs for C_n such that the weight has multiplicity one. Therefore $K_{\zeta,\mu_2} \neq 1$. And finally we conclude that there are no dominant weights ζ with $(\zeta; \mu_2)$ a nonprimitive pair and $K_{\zeta,\mu_2} = 1$. ☐

5 Nonprimitive pairs for D_n, $n \geq 4$

In this case recall that $\mu_1 = \mu_2 = \mu = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ is the highest short and long root since all roots are the same length. We find all dominant weights ζ such that $\zeta \succ \mu$ and $(\zeta; \mu)$ is not a primitive pair. Then we calculate $K_{\zeta,\mu}$.

Lemma 5.1. *For $\mathfrak{g}$ of type D_n, the only highest weights ζ such that $\zeta \succ \mu$, $(\zeta; \mu)$ is not a primitive pair and $K_{\zeta,\mu} = 1$ are*

1. *for $n = 4$*
   $\zeta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$
   $\zeta_2 = \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4$
   $\zeta_3 = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$

2. *for $n \geq 5$ $\zeta_4 = 2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$*

Proof. First we consider the case $n = 4$ and then show the general result.
Case n = 4

In this case $\mu = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. The weight $\zeta = m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + m_4\alpha_4$ is a dominant weight if and only if the following inequalities hold.

$$\begin{aligned} m_2 &\leq 2m_1 & (1)\\ m_1 + m_3 + m_4 &\leq 2m_2 & (2)\\ m_2 &\leq 2m_3 & (3)\\ m_2 &\leq 2m_4 & (4) \end{aligned}$$

For ζ such that $\zeta \succ \mu$ and $(\zeta;\mu)$ is not a primitive pair we have (*) $m_1 \geq 1$, $m_2 \geq 2$, $m_3 \geq 1$ and $m_4 \geq 1$. Hence at least one of the following must hold: (a) $m_1 = 1$, (b) $m_2 = 2$, (c) $m_3 = 1$ or (d) $m_4 = 1$. We consider each case.

(a) If $m_1 = 1$, then $m_2 = 2$ by (1) and (*) and $m_3 + m_4 \leq 3$ by (2). Hence we have one of the following three cases: $m_3 = m_4 = 1$, $m_3 = 2$ and $m_4 = 1$ or $m_3 = 1$ and $m_4 = 2$. The corresponding weights are respectively $\zeta = \mu$, which is ruled out, $\zeta = \zeta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ and $\zeta = \zeta_2 = \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4$, respectively.

(b) If $m_2 = 2$ then $m_1 = 1$ or $m_1 = 2$ since $m_1 \leq 2$ by (2). If $m_1 = 1$, we have the previous case, so let $m_1 = 2$. Then by (2), $m_3 = m_4 = 1$ and $\zeta = \zeta_3 = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

(c) and (d) If either $m_3 = 1$ or $m_4 = 1$, then $m_2 = 2$ by (3) or (4) respectively and (*). Then we have ζ as in the previous cases.

Thus for $n = 4$ we have 3 dominant weights ζ such that $(\zeta;\mu)$ is not a primitive pair:

$$\begin{aligned} \zeta = \zeta_1 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4\\ \zeta = \zeta_2 &= \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4\\ \zeta = \zeta_3 &= 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \end{aligned}$$

In each case the difference $\zeta_i - \mu$ yields $S = \{\alpha_i\}$ and $g(S) \cong A_1$. Then $p(\xi_1) = 2\alpha_1$ and $p(\mu) = \alpha_1$. By Lemma 1.5 we conclude that $K_{\zeta_i,\mu} = 1$ in each case.

Case $\mathbf{n \geq 5}$

For $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$ a dominant weight in this case the following inequalities must hold:

$$\begin{aligned} m_2 &\leq 2m_1 & (1)\\ m_{i-1} + m_{i+1} &\leq 2m_i,\ i = 2, \ldots, n-3 & (2)\\ m_{n-3} + m_{n-1} + m_n &\leq 2m_{n-2} & (3)\\ m_{n-2} &\leq 2m_{n-1} & (4)\\ m_{n-2} &\leq 2m_n & (5) \end{aligned}$$

We show that the following are the only dominant weights $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$ such that $\zeta \succ \mu$ and $(\zeta;\mu)$ is a nonprimitive pair. Then we consider $K_{\zeta,\mu}$ in

each case.

$$\begin{aligned}
\zeta = \zeta_1 &= 2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\
\zeta = \zeta_2 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + m_4\alpha_4 + \cdots + m_n\alpha_n, \text{ with } m_i \geq 3,\ i = 4, \ldots, n-2, \\
&\quad \text{and } m_i \geq 2,\ i = n-1, n
\end{aligned}$$

Note that a dominant weight ζ such that $\zeta \succ \mu$ and $(\zeta;\mu)$ is a nonprimitive pair must satisfy (*) $m_i \geq 1$ for $i = 1, n-1, n$ and $m_i \geq 2$ for $i = 2, \ldots, n-2$ with at least one of these inequalities an equality for $i = 1, \ldots, n$. We consider each case below.

Lemma 5.0.5. *Suppose that $m_i = 2$ for some i, $i = 3, \ldots, n-2$, then $m_i = 2$ for all $i = 2, \ldots, n-2$.*

Proof. Suppose that $m_i = 2$ for some $i = 3, \ldots, n-3$. Then by (2), $m_{i-1} + m_{i+1} \leq 4$, but $m_{i-1} \geq 2$ and $m_{i+1} \geq 2$ by (*) and therefore $m_{i-1} = m_{i+1} = 2$. By induction on (2), $m_i = 2$ for $i = 2, \ldots, n-2$. If $m_{n-2} = 2$, then by (*) and (3) $2 + m_{n-1} + m_n \leq m_{n-3} + m_{n-1} + m_n \leq 2m_{n-2} = 4$. This implies that $m_{n-1} = m_n = 1$ and $m_{n-3} = 2$. By the previous case $m_i = 2$ for $i = 2, \ldots, n-2$. ◻

Case $m_{n-1} = 1$ or $m_n = 1$.

If $m_{n-1} = 1$, then by (4) and (*), $m_{n-2} = 2$ and by (3) and (*), we have $m_{n-3} = 2$ and $m_n = 1$. Similarly, if $m_n = 1$, then we conclude that $m_{n-1} = 1$ and $m_{n-3} = 2$. It now follows from Sublemma 5.0.5 that if $m_{n-1} = 1$ or $m_n = 1$ then $m_{n-1} = m_n = 1$ and $m_i = 2$ for $i = 2, \ldots, n-2$.

From (2) we see that $m_1 \leq 2$ since $m_1 + 2 = m_1 + m_3 \leq 2m_2 = 4$. Hence either $m_1 = 1$, in which case $\zeta = \mu$, which is ruled out, or $m_1 = 2$ for which $\zeta = \zeta_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. Then ζ_1 is a dominant weight such that $(\zeta_1;\mu)$ is not a primitive pair.

Case $m_i = 2$ for some $i = 3, \ldots, n-2$.

Then by Sublemma 5.0.5, $m_i = 2$ for all $i = 2, \ldots, n-2$. By (3) $m_{n-1} = m_n = 1$ and we have the same result as in the previous case.

Case $m_2 = 2$.

If $m_3 = 2$ also, then we have the previous case, so we may assume that $m_i \geq 3$ for $i = 3, \ldots, n-2$. We show that there exists a dominant weight ζ_2 distinct from the previous weight ζ_1 such that $(\zeta_2;\mu)$ is nonprimitive. By (4) $3 \leq m_{n-2} \leq 2m_{n-1}$ and by (5) $3 \leq m_{n-2} \leq 2m_n$, giving $m_{n-1} \geq 2$ and $m_n \geq 2$. By (2) we have $m_1 + m_3 \leq 2m_2 = 4$, but we also know that $m_3 \geq 3$ and therefore $m_1 = 1$ and $m_3 = 3$. Thus $\zeta_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + m_4\alpha_4 + \cdots + m_n\alpha_n$ is a dominant weight such that $(\zeta_2;\mu)$ is not a primitive pair if $m_i \geq 3$ for $i = 4, \ldots, n-2$ and $m_i \geq 2$ for $i = n-1, n$.

Case $m_1 = 1$.

By (1) and (*) $m_2 = 2$ and we have the same result as in the previous case.

Thus the following are the only 2 dominant weights ζ such that $(\zeta;\mu)$ is not a primitive pair:

$$\begin{aligned}\zeta = \zeta_1 &= 2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\ \zeta = \zeta_2 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + m_4\alpha_4 + \cdots + m_n\alpha_n, \text{ with } m_i \geq 3,\ i = 4, \ldots, n-2, \\ &\quad \text{and } m_i \geq 2,\ i = n-1, n\end{aligned}$$

We find $K_{\zeta_i,\mu}$ in each case.

From the difference $\zeta_1 - \mu = \alpha_1$, we see that $S = \{\alpha_1\}$ and $\mathfrak{g}(S) \cong A_1$. Then $p(\zeta_1) = 2\alpha_1$ and $p(\mu) = \alpha_1$ and by Lemma 1.5, $K_{\zeta_1,\mu} = 1$. Note: this is ζ_4 of Lemma 5.1, which is the same as ζ_3 for $n = 4$.

Next we consider ζ_2, first for the case $n = 5$. In this case, $\zeta_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + m_4\alpha_4 + m_5\alpha_5$, where $m_4 \geq 2$ and $m_5 \geq 2$, and $\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$. The inequality (3) gives $2 + m_4 + m_5 \leq 2m_3 = 6$ and we conclude then that the only possible values for m_4 and m_5 are $m_4 = m_5 = 2$. In this case $\zeta_2 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5$.

From the difference $\zeta_2 - \mu = \alpha_3 + \alpha_4 + \alpha_5$ we see that $S = \{\alpha_3, \alpha_4, \alpha_5\}$ and through the relabeling $\{\alpha_3, \alpha_4, \alpha_5\}$ as $\{\alpha_2, \alpha_1, \alpha_3\}$ we find $\mathfrak{g}(S) \cong A_3$. This yields $p(\zeta_2) = 2\alpha_1 + 3\alpha_2 + 2\alpha_3$ and $p(\mu) = \alpha_1 + 2\alpha_2 + \alpha_3$. We see that $(p(\zeta_2); p(\mu))$ is a primitive pair and we can now apply Theorem 1.2 to determine if $K_{\zeta_2,\mu} = K_{p(\zeta_2),p(\mu)} = 1$. According to this result, if $K_{p(\zeta_2),p(\mu)} = 1$, then $p(\zeta_2) = l\omega_1$. Hence $p(\zeta_2) = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 = \frac{l}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3)$ for some positive integer l. However, there is no such l and therefore $K_{\zeta_2,\mu} \neq 1$.

Next, consider $n \geq 6$. From the difference $\zeta_2 - \mu = \alpha_3 + (m_4 - 2)\alpha_4 + \cdots + (m_n - 1)\alpha_n$ we observe that $S = \{\alpha_3, \ldots, \alpha_n\}$ and since $n \geq 6$, the Dynkin diagram gives $\mathfrak{g}(S) \cong D_{n-2}$. Relabeling $\{\alpha_3, \ldots, \alpha_n\}$ as $\{\alpha_1, \ldots, \alpha_{n-2}\}$ we obtain $p(\zeta_2) = 3\alpha_1 + m_4\alpha_2 + \cdots + m_n\alpha_{n-2}$ and $p(\mu) = 2\alpha_1 + \cdots + 2\alpha_{n-4} + \alpha_{n-3} + \alpha_{n-2}$. We see that $(p(\zeta_2); p(\mu))$ is a primitive pair in D_{n-2}, but by Theorem 1.2 we also observe that there are no primitive pairs $(\zeta;\mu)$ in D_{n-2} such that $K_{\zeta,\mu} = 1$, therefore $K_{\zeta_2,\mu} \neq 1$. □

6 Nonprimitive pairs for E_n

6.1 Nonprimitive pairs for E_6

In this case, recall that $\mu_1 = \mu_2 = \mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is the highest root.

Lemma 6.1. *For $\mathfrak{g}$ of type E_6 there are no highest weights ζ with $\zeta \succ \mu$ and $(\zeta;\mu)$ a nonprimitive pair such that $K_{\zeta,\mu} = 1$.*

Proof. Recall that any dominant weight $\zeta = m_1\alpha_1 + \cdots + m_6\alpha_6$ must satisfy

the following:

$$
\begin{aligned}
m_3 &\leq 2m_1 & (1)\\
m_4 &\leq 2m_2 & (2)\\
m_1 + m_4 &\leq 2m_3 & (3)\\
m_2 + m_3 + m_5 &\leq 2m_4 & (4)\\
m_4 + m_6 &\leq 2m_5 & (5)\\
m_5 &\leq 2m_6 & (6)
\end{aligned}
$$

Note that $\zeta \succ \mu$ yields the following inequalities for $\zeta = m_1\alpha_1 + \cdots + m_6\alpha_6$: (*) $m_i \geq 1$ for $i = 1, 6$, $m_i \geq 2$ for $i = 2, 3, 5$ and $m_4 \geq 3$. In addition, if ζ is a dominant weight such that $(\zeta; \mu)$ is not a primitive pair, at least one of the inequalities will be an equality for some i. We consider each of these cases.

Case 1 Suppose that $m_1 = 1$. By (1) and (*) $m_3 = 2$ and by (3) $1 + m_4 \leq 4$, thus by (*), $m_4 = 3$. By (4), $m_2 + 2 + m_5 \leq 2m_4 = 6$ which with (*) implies that $m_2 = m_5 = 2$. Lastly, by (5), $m_6 = 1$. Thus if $m_1 = 1$, then $\zeta = \mu$, which is ruled out.

Case 2 Next, suppose that $m_2 = 2$. By (2) $m_4 \leq 4$, so $m_4 = 3$ or $m_4 = 4$ by (*). If $m_4 = 3$, then by (4) and (*) we conclude that $m_2 = m_3 = m_5 = 2$, which implies that $m_1 = 1$ by (3). Again, $m_6 = 1$ by (5) and we find that $\zeta = \mu$, which is ruled out.

Let $m_4 = 4$. By (3) we have $m_1 + 4 \leq 2m_3$ and by (5) $4 + m_6 \leq 2m_5$ which give $m_3 \geq 3$ and $m_5 \geq 3$. Inequality (4) yields $m_3 + m_5 \leq 6$ and thus we conclude that $m_3 = m_5 = 3$. Note that $m_1 = 2$ follows from (1) and (3). Also, $m_6 = 2$ by (5) and (6). Thus $\zeta = \zeta_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6$ is a dominant weight such that $(\zeta_1; \mu)$ is not a primitive pair.

Case 3 Suppose that $m_3 = 2$. By (3) and (*) $m_1 = 1$ and $m_4 = 3$. Therefore we are in the first case considered and $\zeta = \mu$, which we have ruled out.

Case 4 Suppose that $m_4 = 3$. Then by (2), $m_2 \geq 2$, but by (4) and (*), $m_2 = m_3 = m_5 = 2$. As seen above, if $m_2 = 2$, then $\zeta = \mu$ or $\zeta = \zeta_1$, however the identity $m_4 = 3$ yields only $\zeta = \mu$, which is ruled out.

Case 5 Suppose that $m_5 = 2$. By (5) $m_4 + m_6 \leq 4$, and thus $m_4 = 3$ and $m_6 = 1$ by (*). We are now in the previous case.

Case 6 Suppose that $m_6 = 1$. Then by (6) and (*) $m_5 = 2$ and we are in the previous case.

From the above, we conclude that $\zeta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6$ is the only dominant weight such that $(\zeta; \mu)$ is a nonprimitive pair. We will show that $K_{\zeta,\mu} \neq 1$.

From the difference $\zeta - \mu = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ we observe that $S = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and by considering the Dynkin diagram of E_6, we see that $\mathfrak{g}(S) \cong A_5$. Relabeling $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ as $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, we obtain $p(\zeta) = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5$ and $p(\mu) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$. We notice that $(p(\zeta); p(\mu))$ is a primitive pair and therefore we apply Theorem 1.2 to determine if $K_{p(\zeta),p(\mu)} = 1$. In the A_5 case, if $K_{p(\zeta),p(\mu)} = 1$, then

$p(\zeta) = l\omega_1 = \frac{l}{6}(5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5)$ for some positive integer l. Clearly, there is no such l and therefore $K_{\zeta,\mu} \neq 1$. $\square$

6.2 Nonprimitive pairs for E_7

In this case, recall that $\mu_1 = \mu_2 = \mu = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ is the highest root.

Lemma 6.2. *For $\mathfrak{g}$ of type E_7 there are no highest weights ζ with $\zeta \succ \mu$ and $(\zeta;\mu)$ a nonprimitive pair such that $K_{\zeta,\mu} = 1$.*

Proof. Recall that any dominant weight $\zeta = m_1\alpha_1 + \cdots + m_7\alpha_7$ must satisfy the following:

$$\begin{aligned} m_3 &\leq 2m_1 && (1)\\ m_4 &\leq 2m_2 && (2)\\ m_1 + m_4 &\leq 2m_3 && (3)\\ m_2 + m_3 + m_5 &\leq 2m_4 && (4)\\ m_4 + m_6 &\leq 2m_5 && (5)\\ m_5 + m_7 &\leq 2m_6 && (6)\\ m_6 &\leq 2m_7 && (7) \end{aligned}$$

Note that $\zeta \succ \mu$ yields the following inequalities for $\zeta = m_1\alpha_1 + \cdots + m_7\alpha_7$: (*) $m_i \geq 2$ for $i = 1, 2, 6$, $m_i \geq 3$ for $i = 3, 5$, $m_4 \geq 4$ and $m_7 \geq 1$. In addition, if ζ is a dominant weight such that $(\zeta;\mu)$ is not a primitive pair, at least one of the inequalities will be an equality for some i. We consider each of these cases.

We show that the inequalities above can be satisfied for $\zeta \succ \mu$, $(\zeta;\mu)$ a nonprimitive pair, only if $\zeta = \zeta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7$ or $\zeta = \zeta_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7$. We then show that $K_{\zeta_i,\mu} \neq 1$ for $i = 1, 2$.

Case 1 Suppose that $m_1 = 2$. By (1) and (*) $m_3 = 3$ or $m_3 = 4$.

First suppose that $m_3 = 3$. By (3) $2 + m_4 \leq 6$ and combined with (*), we conclude $m_4 = 4$. Inequality (4) then gives $m_2 + 3 + m_5 \leq 8$ and since $m_2 \geq 2$ and $m_5 \geq 3$ by (*), these are in fact equalities. From (5) and (*), we see that $m_6 = 2$ which means that $m_7 = 1$ by (6). Thus if $m_1 = 2$ and $m_3 = 3$, then $\zeta = \mu$, which is ruled out.

Now let $m_3 = 4$. By inequality (3) $2 + m_4 \leq 8$, giving $m_4 \leq 6$. By (*) $m_4 \geq 4$. We consider each case $m = 4, 5, 6$ individually.

$\mathbf{m_4 = 4}$ Inequality (4) and (*) give $2 + 4 + 3 \leq m_2 + m_3 + m_5 \leq 2m_4 = 8$, an obvious contradiction. Therefore, $m_4 \neq 4$.

$\mathbf{m_4 = 5}$ By (2) $5 = m_4 \leq 2m_2$ and thus $m_2 \geq 3$. Also, by (5) and (*), $5 + 2 \leq m_4 + m_6 \leq 2m_5$, so $m_5 \geq 4$. We get a contradiction by (4) since then $3 + 4 + 4 \leq m_2 + m_3 + m_5 \leq 2m_4 = 10$. Therefore, $m_4 \neq 5$.

$\mathbf{m_4 = 6}$ We consider the inequalities in the case $m_4 = 6$ where $m_3 = 4$ and $m_1 = 2$. We show that $\zeta = \zeta_1$ or $\zeta = \zeta_2$ as listed above.

(a) $m_2 \geq 3$
This follows from (2) since $m_4 = 6$.

(b) $m_5 = 5$
From (a) and (4) we have $7+m_5 \leq m_2+m_3+m_5 \leq 2m_4 = 12$. Hence $m_5 \leq 5$. If $m_5 \leq 4$, then by (5) we have $6+m_6 = m_4+m_6 \leq 2m_5 \leq 8$, which implies $m_6 \leq 2$. By (*) $m_6 \geq 2$, and hence $m_6 = 2$ and $m_5 = 4$. By (6) we obtain $4+m_7 = m_5+m_7 \leq 2m_6 = 4$, which is impossible. Hence $m_5 = 5$.

(c) $m_2 = 3$
From (4) and (b) we have $m_2 + 9 = m_2 + m_3 + m_5 \leq 2m_4 = 12$. Hence $m_2 \leq 3$ and equality holds by (a).

(d) $m_6 = 4$
From (5) and (b) we have $6 + m_6 = m_4 + m_6 \leq 2m_5 = 10$, which implies $m_6 \leq 4$. If $m_6 \leq 3$, then by (6) and (b) we have $5 + m_7 = m_5 + m_7 \leq 2m_6 \leq 6$. This implies $m_7 \leq 1$, and equality holds by (*). This implies that $m_6 = 3$, but by (7) we have $m_6 \leq 2m_7 = 2$. This contradiction shows that $m_6 = 4$.

(e) $m_7 = 2$ or 3
By (7) and (d) we have $4 = m_6 \leq 2m_7$, which implies that $m_7 \geq 2$. By (6), (b) and (d) we have $5 + m_7 = m_5 + m_7 \leq 2m_6 = 8$, which implies that $m_7 \leq 3$.

We then have 2 dominant weights ζ_i such that $(\zeta_i; \mu)$ are not primitive pairs: $\zeta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7$ and $\zeta_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7$.

Case 2 Suppose that $m_2 = 2$. Then by (2) and (*) $4 \leq m_4 \leq 2m_2 = 4$, so $m_4 = 4$. By (4) and (*), $2+3+3 \leq m_2+m_3+m_5 \leq 2m_4 = 8$, which results in $m_3 = m_5 = 3$. Then by (3) and (*) $2+4 \leq m_1+m_4 \leq 2m_3 = 6$ and therefore $m_1 = 2$. We are now in Case 1 with $m_3 = 3$, but this was ruled out.

Case 3 Suppose that $m_3 = 3$. By (3) and (*) $2+4 \leq m_1+m_4 \leq 2m_3 = 6$, and hence $m_1 = 2$ and $m_4 = 4$. We are again in Case 1 with $m_3 = 3$, which was ruled out.

Case 4 Suppose that $m_4 = 4$. By (4) and (*) $2+3+3 \leq m_2+m_3+m_5 \leq 2m_4 = 8$, yielding $m_2 = 2$, $m_3 = 3$, and $m_5 = 3$. We are now in Case 2, which was ruled out.

Case 5 Let $m_5 = 3$. By (5) and (*) $4+2 \leq m_4+m_6 \leq 2m_5 = 6$, and hence $m_4 = 4$ and $m_6 = 2$. We are now in Case 4, which was ruled out.

Case 6 Let $m_6 = 2$. Then by (6) and (*) $3+1 \leq m_5+m_7 \leq 2m_6 = 4$, yielding the equalities $m_5 = 3$ and $m_7 = 1$. We are in Case 5, which was ruled out.

Case 7 Let $m_7 = 1$. By (7) and (*) $2 \leq m_6 \leq 2m_7 = 2$ which implies that $m_6 = 2$. Again, we fall into the previous case which was ruled out.

Thus, the only two dominant weights ζ such that $\zeta \succ \mu$ and $(\zeta; \mu)$ is not a primitive pair are ζ_1 and ζ_2 as in Case 1. We now consider $K_{\zeta_i,\mu}$ in each case.

From the difference $\zeta_1 - \mu = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$ we observe that $S = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ and by considering the Dynkin diagram of E_7, we see that $\mathfrak{g}(S) \cong D_6$. Then relabeling $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ as $\{\alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}$, we obtain $p(\zeta_1) = 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6$ and $p(\mu) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6$. Then $(p(\zeta_1); p(\mu))$ is a primitive pair for D_6. By Theorem 1.2, for type D_6, there are no primitive pairs such that the dimension of the weight space is one, so therefore $K_{\zeta_1,\mu} \neq 1$.

The case of ζ_2 is similar to the previous one. From the difference $\zeta_2 - \mu = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7$ we observe that again $S = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ and then $\mathfrak{g}(S) \cong D_6$. Then with the same relabeling $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ as $\{\alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}$, we obtain $p(\zeta_2) = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6$ and $p(\mu) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6$. Then $(p(\zeta_2); p(\mu))$ is a primitive pair for D_6, and by Theorem 1.2, we conclude that $K_{\zeta_2,\mu} \neq 1$. $\square$

6.3 Nonprimitive pairs for E_8

In this case, recall that $\mu_1 = \mu_2 = \mu = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Lemma 6.3. *For $\mathfrak{g}$ of type E_8 there are no highest weights $\zeta \succ \mu$ such that $(\zeta; \mu)$ is a nonprimitive pair with $K_{\zeta,\mu} = 1$.*

Proof. Recall that any dominant weight $\zeta = m_1\alpha_1 + \cdots + m_8\alpha_8$ must satisfy the following:

$$\begin{aligned} m_3 &\leq 2m_1 && (1)\\ m_4 &\leq 2m_2 && (2)\\ m_1 + m_4 &\leq 2m_3 && (3)\\ m_2 + m_3 + m_5 &\leq 2m_4 && (4)\\ m_4 + m_6 &\leq 2m_5 && (5)\\ m_5 + m_7 &\leq 2m_6 && (6)\\ m_6 + m_8 &\leq 2m_7 && (7)\\ m_7 &\leq 2m_8 && (8) \end{aligned}$$

Note that $\zeta \succ \mu$ yields the following inequalities for $\zeta = m_1\alpha_1 + \cdots + m_8\alpha_8$: (*) $m_i \geq 2$ for $i = 1, 8$, $m_i \geq 3$ for $i = 2, 7$, $m_i \geq 4$ for $i = 3, 6$, $m_4 \geq 6$ and $m_5 \geq 5$. In addition, if ζ is a dominant weight such that $(\zeta; \mu)$ is not a primitive pair, at least one of the inequalities will be an equality for some i. We consider each of these cases.

Case 1 Let $m_1 = 2$. By (1) $m_3 \leq 2m_1 = 4$ and by (*) $m_3 \geq 4$, so we conclude that $m_3 = 4$. By (3) and (*) it then follows that $2 + 6 \leq m_1 + m_4 \leq 2m_3 = 8$, resulting in $m_4 = 6$. We find $m_2 = 3$ and $m_5 = 5$ by (*) and (4). From (5) and (*) we obtain $m_6 = 4$. From (6) and (*) we see that $m_7 = 3$. By (7) and (*) we obtain $m_8 = 2$. Hence $m_1 = 2$ implies that $\zeta = \mu$, which is ruled out.

Case 2 Let $m_2 = 3$. By (2) and (*) $m_4 = 6$. By (4) and (*) $m_3 = 4$ and $m_5 = 5$. By (3) and (*) we have $m_1 = 2$. By the first case $\zeta = \mu$, which is ruled out.
Case 3 Let $m_3 = 4$. Then (3) and (*) force $m_1 = 2$ and $m_4 = 6$. By Case 1 we have $\zeta = \mu$, which is ruled out.
Case 4 Let $m_4 = 6$. Then (4) and (*) yield $m_2 = 3$, $m_3 = 4$ and $m_5 = 5$. By Case 2, we have $\zeta = \mu$, which is ruled out.
Case 5 Let $m_5 = 5$. By (5) and (*) $m_4 = 6$ and $m_6 = 4$. By the previous case $\zeta = \mu$, which is ruled out.
Case 6 Let $m_6 = 4$. By (6) and (*) $m_5 = 5$ and $m_7 = 3$. We are now in Case 5.
Case 7 Let $m_7 = 3$. By (7) and (*) $m_6 = 4$ and $m_8 = 2$. We are now in Case 6.
Case 8 Let $\zeta \succ \mu$ be a dominant weight with $(\zeta;\mu)$ a nonprimitive pair and $m_8 = 2$. Since the previous seven cases have been ruled out we may assume that $m_1 \geq 3$, $m_2 \geq 4$, $m_3 \geq 5$, $m_4 \geq 7$, $m_5 \geq 6$, $m_6 \geq 5$ and $m_7 \geq 4$. It follows that $(p(\zeta);p(\mu))$ is a primitive pair, where $S = \{\alpha_1, \ldots, \alpha_7\}$. Hence $\mathfrak{g}(S) \cong E_7$ by an inspection of the Dynkin diagram. However by Theorem 1.2 no primitive pairs $(p(\zeta);p(\mu))$ with $K_{p(\zeta),p(\mu)} = 1$ exist for E_7.

Remark With further work one can show that the only dominant weight ζ with $\zeta \succ \mu$, $m_8 = 2$ and $(\zeta;\mu)$ a nonprimitive pair is $\zeta = 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8$. $\square$

7 Nonprimitive pairs for F_4

Recall that in this case the highest short and long roots are $\mu_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ and $\mu_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$.

Lemma 7.1. *For $\mathfrak{g}$ of type F_4 there are no highest weights ζ such that $\zeta \succ \mu_2$, $(\zeta;\mu_i)$ is a nonprimitive pair and $K_{\zeta,\mu_i} = 1$ for $i = 1, 2$.*

Proof. Recall that a weight ζ is a dominant weight if and only if the following inequalities hold:

$$\begin{aligned} m_2 &\leq 2m_1 & (1)\\ m_1 + m_3 &\leq 2m_2 & (2)\\ 2m_2 + m_4 &\leq 2m_3 & (3)\\ m_3 &\leq 2m_4 & (4) \end{aligned}$$

Since $\zeta \succ \mu_2$ the following inequalities must hold: (*) $m_1 \geq 2$, $m_2 \geq 3$, $m_3 \geq 4$ and $m_4 \geq 2$. In addition if ζ is a dominant weight such that the pair is not primitive, at least one of the inequalities must be an equality. We consider each case.
Case 1 Let $m_1 = 2$. Then by (1) and (*) $3 \leq m_2 \leq 4$.

(a) If $m_2 = 3$, then by (2) and (*) $m_3 = 4$. Then by (3) and (*) $m_4 = 2$. Hence $\zeta = \mu_2$, which is ruled out.

(b) If $m_2 = 4$, then by (2) $m_3 \leq 6$. By (3) $m_3 \geq 5$. Therefore $m_3 = 5$ or $m_3 = 6$. If $m_3 = 5$ then by (3) $m_4 = 2$, resulting in a contradiction in inequality(4): $5 = m_3 \leq 2m_4 = 4$. Hence $m_3 = 6$. Inequality (3) implies that $m_4 \leq 4$. By (4) $6 = m_3 \leq 2m_4$, which implies that $m_4 = 3$ or $m_4 = 4$.

Thus in Case 1 there are two dominant weights ζ_i such that $\zeta_i \succ \mu_2$ and $(\zeta_i; \mu_2)$ is not a primitive pair, namely $\zeta_1 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4$ and $\zeta_2 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4$.
Case 2 Let $m_2 = 3$. Then by (2) and (*) $m_1 = 2$ and $m_3 = 4$. We are in (a) of Case 1, which is ruled out.
Case 3 Let $m_3 = 4$. Then by (3) and (*) $m_2 = 3$ and $m_4 = 2$. By (2) and (*) $m_1 = 2$ and $\zeta = \mu_2$, which is ruled out.
Case 4 Let $m_4 = 2$. By (4) and (*) we have $4 \leq m_3 \leq 2m_4 = 4$ and thus $m_3 = 4$ as in the previous case, forcing $\zeta = \mu_2$, which is ruled out.

Thus we have found two dominant weights ζ_i such that $(\zeta_i; \mu_2)$ is a nonprimitive pair:

$$\begin{aligned} \zeta_1 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 \\ \zeta_2 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 \end{aligned}$$

Next we show that $K_{\zeta_i,\mu_2} \neq 1$ in each case. Note that by Theorem 1.2 $K_{\zeta_i,\mu_1} \neq 1$ since $(\zeta_i; \mu_1)$ is a primitive pair for $i = 1, 2$.

From the difference $\zeta_1 - \mu_2 = \alpha_2 + 2\alpha_3 + \alpha_4$ we observe that $S = \{\alpha_2, \alpha_3, \alpha_4\}$ and by comparing Dynkin diagrams that $\mathfrak{g}(S) \cong C_3$. Relabeling $\{\alpha_2, \alpha_3, \alpha_4\}$ as $\{\alpha_3, \alpha_2, \alpha_1\}$ yields $p(\zeta_1) = 3\alpha_1 + 6\alpha_2 + 4\alpha_3$ and $p(\mu_2) = 2\alpha_1 + 4\alpha_2 + 3\alpha_3$. Now $(p(\zeta_1); p(\mu_2))$ is a primitive pair in C_3 and by Theorem 1.2 there are no primitive pairs for C_3 such that the weight space has dimension one. Therefore $K_{\zeta_1,\mu_2} \neq 1$.

From the difference $\zeta_2 - \mu_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4$ we observe that again $S = \{\alpha_2, \alpha_3, \alpha_4\}$ and $\mathfrak{g}(S) \cong C_3$. Relabeling $\{\alpha_2, \alpha_3, \alpha_4\}$ as $\{\alpha_3, \alpha_2, \alpha_1\}$ yields $p(\zeta_2) = 4\alpha_1 + 6\alpha_2 + 4\alpha_3$ and $(\mu_2) = 2\alpha_1 + 4\alpha_2 + 3\alpha_3$. Now $(p(\zeta_2); p(\mu_2))$ is a primitive pair in C_3 and similarly by Theorem 1.2 we conclude that $K_{\zeta_1,\mu_2} \neq 1$.

Thus there are no dominant weights ζ such that $(\zeta; \mu)$ is a primitive pair and $K_{\zeta,\mu} = 1$ for F_4. □

8 Nonprimitive pairs for G_2

Recall that the highest short and long roots in this case are $\mu_1 = 2\alpha_1 + \alpha_2$ and $\mu_2 = 3\alpha_1 + 2\alpha_2$.

Lemma 8.1. *For $\mathfrak{g}$ of type G_2, the only highest weight ζ such that $\zeta \succ \mu_2$ and $(\zeta; \mu_i)$ is nonprimitive for $i = 1$ or 2 is $\zeta = 4\alpha_1 + 2\alpha_2$. In this case $(\zeta; \mu_1)$ is primitive and $(\zeta; \mu_2)$ is nonprimitive with $K_{\zeta,\mu_2} = 1$.*

Proof. For $\zeta = m_1\alpha_1 + m_2\alpha_2$ to be a dominant weight, the following must hold:

$$3m_2 \leq 2m_1 \quad (1)$$
$$m_1 \leq 2m_2 \quad (2)$$

We are looking for dominant weights $\zeta \succ \mu_2$ such that $(\zeta;\mu_i)$ is a nonprimitive pair for $i = 1$ or 2. The condition $\zeta \succ \mu_2$ implies that (*) $m_1 \geq 3$ and $m_2 \geq 2$, and hence $(\zeta;\mu_1)$ will be a primitive pair. We then restrict our discussion to finding dominant weights $\zeta \succ \mu_2$ such that $(\zeta;\mu_2)$ is not a primitive pair.

For ζ a dominant weight $\zeta = m_1\alpha_1 + m_2\alpha_2$ such that $(\zeta;\mu_2)$ is not a primitive pair, either $m_1 = 3$ or $m_2 = 2$. If $m_1 = 3$, then $m_2 \geq 2$ by (*) and $m_2 \leq 2$ by (1). Hence $m_2 = 2$ and $\zeta = \mu_2$, which is ruled out. Thus we consider $m_2 = 2$. In this case inequalities (1) and (2) yield either $m_1 = 3$ or $m_1 = 4$. In the first case we again have $\zeta = \mu_2$, which is ruled out, but in the second, we have $\zeta = 4\alpha_1 + 2\alpha_2$, and $(\zeta;\mu_2)$ is not a primitive pair. We consider K_{ζ,μ_2}.

From the difference $\zeta - \mu_2 = \alpha_1$ we see that $S = \{\alpha_1\}$ and then $\mathfrak{g}(S) \cong A_1$. Then $p(\zeta) = 4\alpha_1$ and $p(\mu_2) = 3\alpha_1$. By Theorem 1.2 for a Lie algebra of type A_1, $K_{\zeta,\mu_2} = K_{p(\zeta),p(\mu_2)} = 1$ if $p(\zeta) = l\omega_1$, $p(\mu_2) = a_1\omega_1$ where $a_1 \in \mathbb{Z}_+$ and $(l - a_1) \in 2\mathbb{N}$. In this case, $p(\mu_2) = 3\alpha_1 = 6\omega_1$, so l must be an integer such that $(l-6) \in 2\mathbb{N}$ and $p(\zeta) = l\omega_1 = \frac{l}{2}\alpha_1$. Clearly $l = 8$ satisfies these conditions and therefore $K_{p(\zeta),p(\mu_2)} = K_{\zeta,\mu_2} = 1$. □

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